INTERNATIONAL BACCALAUREATE

Mathematics: applications and interpretation

MAI

EXERCISES [MAI 5.8]

MONOTONY – CONCAVITY – OPTIMIZATION

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A. Paper 1 questions (SHORT)

MONOTONY AND CONCAVITY

1. [Maximum mark: 12]

Differentiate the following functions and hence determine whether each function is increasing or decreasing (or neither).

(i) $f(x) = x^5 + e^x + 1$, (ii) $f(x) = x^3 + \ln x$, $x > 0$

(iii) $f(x) = 5 - 3e^{2x}$ (iv) $f(x) = \dfrac{e^x - 1}{e^x + 1}$

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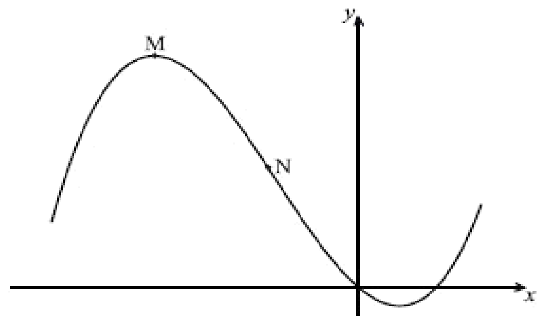
2. [Maximum mark: 8]

Consider $f(x) = \frac{1}{3}x^3 + 2x^2 - 5x$. Part of the graph of f is shown below. There is a maximum point at M, and a point of inflexion at N.



(a) Find $f'(x)$ [2]

(b) Find the x - coordinate of M. [2]

(c) Find the x - coordinate of N. [3]

(d) Write down the equation of the normal line at M. [1]

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3. [Maximum mark: 4]

It is given that $f''(x) = (x-1)(x-3)(x-4)^2$.

Find the points of inflection of f; justify your answer.

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4. [Maximum mark: 5]

Let $f'(x) = -24x^3 + 9x^2 + 3x + 1$.

(a) There are two points of inflexion on the graph of f. Write down the x-coordinates of these points. [3]

(b) Let $g(x) = f''(x)$. Explain why the graph of g has no points of inflexion. [2]

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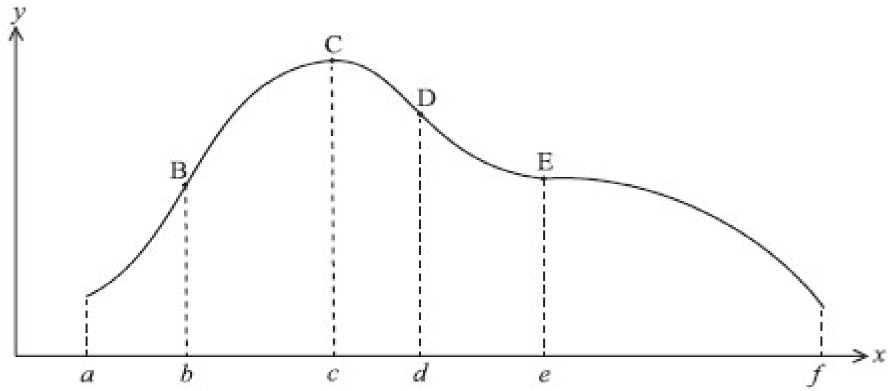
5. [Maximum mark: 8]

The graph of a function *g* is given in the diagram below.



The gradient of the curve has its maximum value at point B and its minimum value at point D. The tangent is horizontal at points C and E.

(a) Complete the table below, by stating whether the first derivative g' is positive or negative, and whether the second derivative g'' is positive or negative.

Interval	g'	g''
$a < x < b$		
$e < x < f$		

[4]

(b) Complete the table below by noting the points on the graph described by the following conditions.

Conditions	Point
$g'(x) = 0, g''(x) < 0$	
$g'(x) < 0, g''(x) = 0$	

[2]

(c) Write down the values of $g'(e)$ and $g''(e)$. [1]

(d) Write down the number of points of inflexion for this curve. [1]

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6. [Maximum mark: 7]

Consider the function $f(x)=\dfrac{3x-2}{2x+5}$.

The graph of this function has a vertical and a horizontal asymptote.

(a) Write down the equations of the asymptotes [2]

(b) Find $f'(x)$, simplifying the answer as much as possible. [2]

(c) Write down the number of stationary points of the graph. Justify your answer. [2]

(d) Write down the number of points of inflection of the graph. [1]

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7. [Maximum mark: 5]

A function f has its first derivative given by $f'(x)=(x-3)^3$.

(a) Find the second derivative. [2]

(b) Find $f'(3)$ and $f''(3)$. [1]

(c) Explain why the point P on the graph with x-coordinate 3 is not a point of inflexion. [2]

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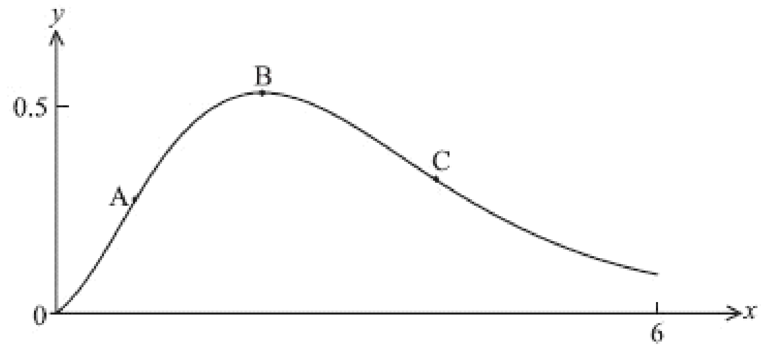
8. [Maximum mark: 8]

The diagram below shows the graph of $f(x)=x^2e^{-x}$ for $0 \le x \le 6$. There are points of inflexion at A and C and there is a maximum at B.



(a) Using the product rule for differentiation, find $f'(x)$. [2]

(b) Find the **exact** value of the ***y*-coordinate** of B. [2]

(c) (i) Show that $f''(x)=(x^2-4x+2)e^{-x}$.

(ii) **Hence**, find the **exact** value of the x- coordinate of C.

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9. [Maximum mark: 6]

Let $g(x)=\frac{\ln x}{x^2}$, for $x>0$. The graph of g has a maximum point at A

(a) Use the quotient rule to show that $g'(x)=\frac{1-2\ln x}{x^3}$. [3]

(b) Find the x-coordinate of A in the form e^a [3]

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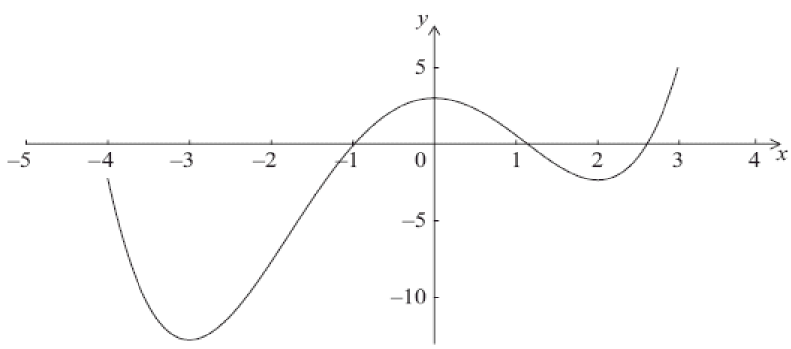
10. [Maximum mark: 6]

A function f is defined for $-4\le x\le 3$. The graph of f is given below.



The graph has a local maximum when $x=0$, and local minima when $x=-3$, $x=2$.

(a) Write down the x-intercepts of the graph of the **derivative** function, f'. [2]

(b) Write down all values of x for which $f'(x)$ is positive. [2]

(c) At point D on the graph of f, the x-coordinate is -0.5. Explain why $f''(x)<0$ at D. [2]

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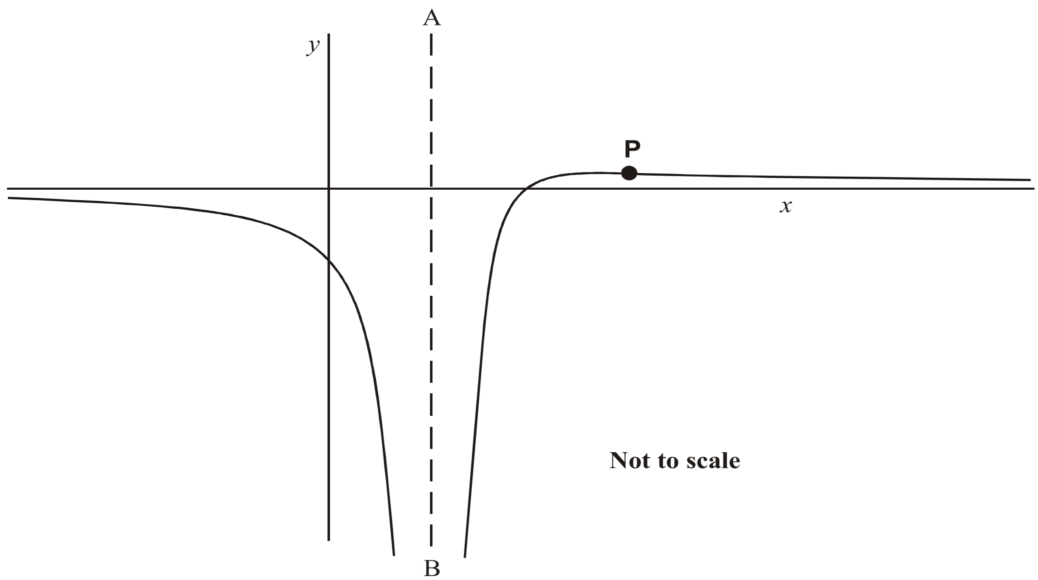
11. [Maximum mark: 8]

Consider the function $h: x \mapsto \dfrac{x-2}{(x-1)^2}, x \neq 1$.

A sketch of part of the graph of h is given below.



The line (AB) is a vertical asymptote. The point P is a point of inflexion.

(a) Write down the **equation** of the vertical asymptote. [1]

(b) Find $h'(x)$ writing your answer in the form $\dfrac{a-x}{(x-1)^n}$ [4]

(c) Given that $h''(x) = \dfrac{2x-8}{(x-1)^4}$, calculate the coordinates of P. [3]

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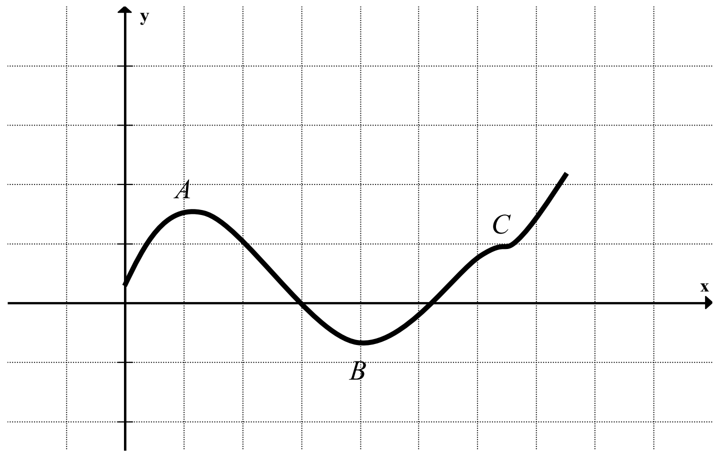
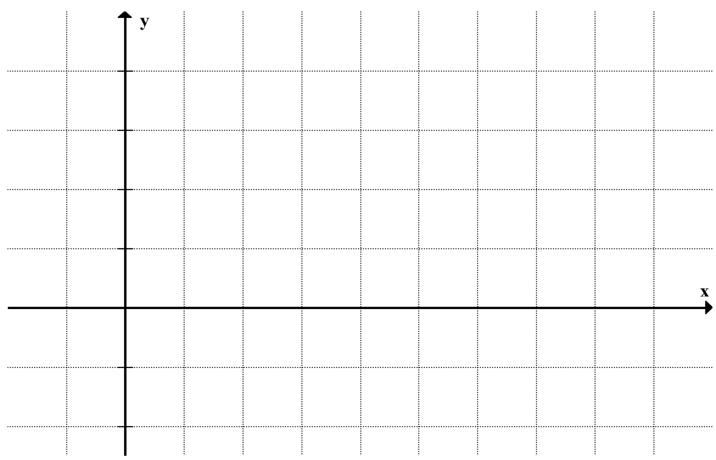
GRAPH OF f'

12. [Maximum mark: 6]

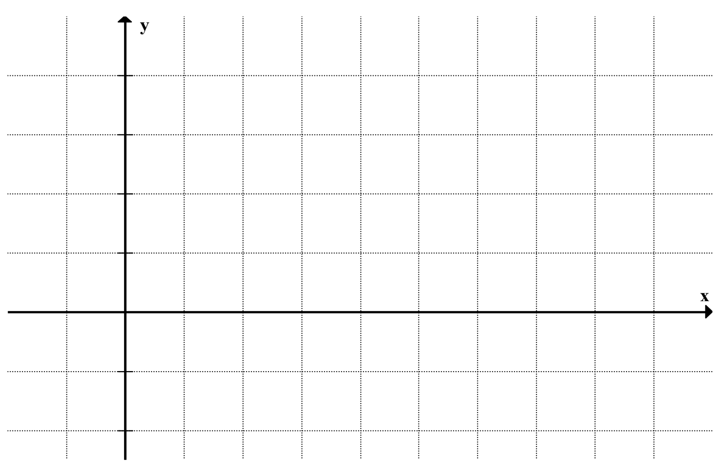
The diagram below shows the graph of $y = f(x)$. The tangent lines on this curve at points *A*, *B*, *C* are parallel to the *x*-axis.



(a) Sketch the graph of the function $y = f'(x)$, by indicating the *x*-intercepts. [3]

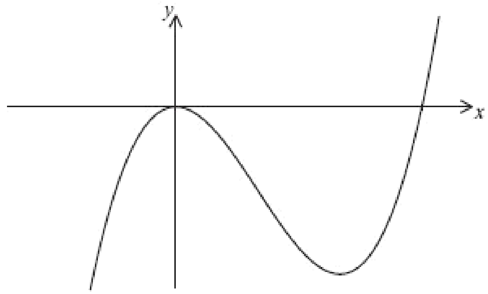


(b) Sketch the graph of the function $y = f''(x)$, by indicating the *x*-intercepts. [3]

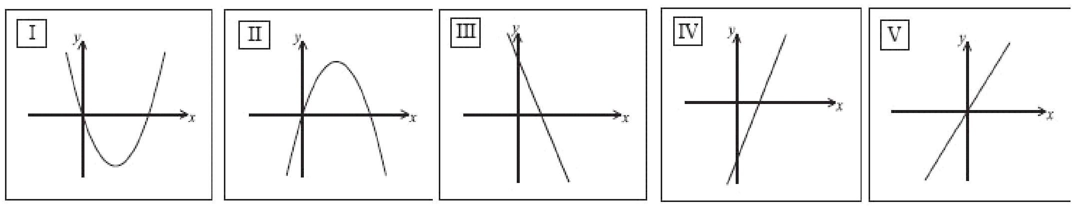


13. [Maximum mark: 4]

The following diagram shows the graph of a function f.



Consider the following diagrams.



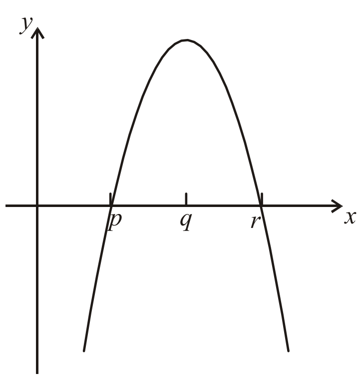
Complete the table below, noting the corresponding diagram for $f'(x)$ and $f''(x)$:

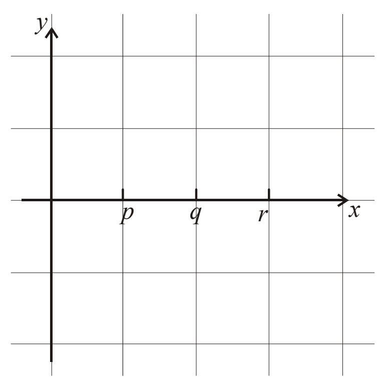
Graph	$f'(x)$	$f''(x)$
Diagram		

14. [Maximum mark: 6]

The first diagram below shows part of the graph of the **gradient** function $y = f'(x)$.

(a) On the grid on the right, sketch a graph of $f''(x)$; indicate the x-intercept.





[2]

(b) Complete the table, for the graph of $y = f(x)$.

	Maximum point on *f*	Inflexion point on *f*
x-coordinate		

[2]

(c) Justify your answer to part (b) (ii). [2]

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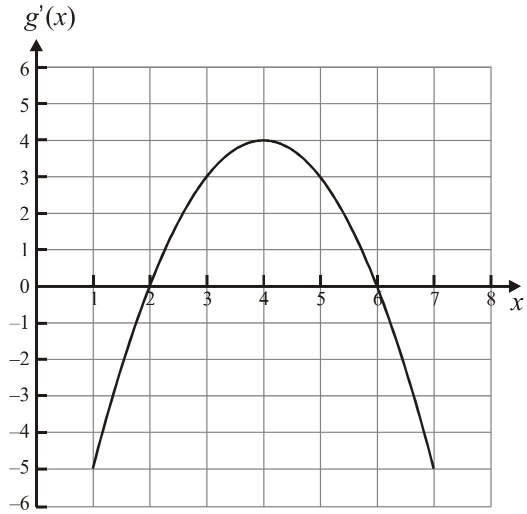
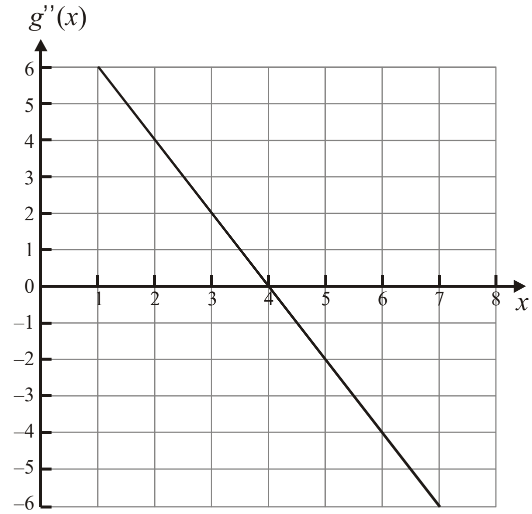
15. [Maximum mark: 8]

Let $y = g(x)$ be a function of x for $1 \le x \le 7$. The graph of g has an inflexion point at P, and a minimum point at M.

Partial sketches of the curves of g' and g'' are shown below.



$y = g'(x)$



$y = g''(x)$

Use the above information to answer the following.

(a) Write down the x-coordinate of P; justify your answer. [2]

(b) Write down the x-coordinate of M; justify your answer. [2]

(c) Given that $g(4) = 0$, sketch the graph of g. Mark the points P and M. [4]

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OPTIMIZATION

16. [Maximum mark: 7]

Consider all rectangles of constant perimeter $4a$.

Find the dimensions of the rectangle of maximum area and **hence** the maximum area.

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17. [Maximum mark: 7]

Consider all rectangles of constant area $A = a^2$.

Find the rectangle of minimum perimeter and justify your answer.

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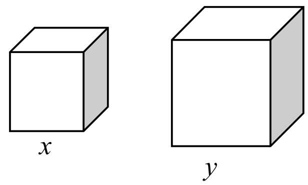
18. [Maximum mark: 9]

Two cubes of edges x and y respectively respectively are shown below.

The total surface area of two cubes is 300 cm^2.



The total surface area of the two cubes is 300 cm^2.

(a) Find an expression of y in terms of x. [3]

(b) Given that there is a minimum value of the total volume of the cubes, find this minimum value and the corresponding dimensions of the two cubes. [6]

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19. [Maximum mark: 9]

A point on the curve $y = x^2$ has coordinates P$\left(a, a^2\right)$. The point A$\left(2, \frac{1}{2}\right)$ does not lie in the curve.

(a) Express the distance D between P and A in terms of a. [2]

(b) Find $\frac{dD}{da}$ [3]

(c) **Hence** find

(i) the coordinates of the point P on the line which is closest to A;

(ii) the minimum distance D between the point A and the curve $y = x^2$. [4]

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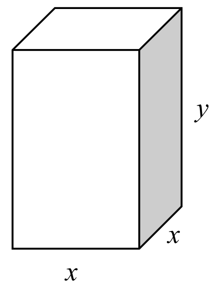
20. [Maximum mark: 9]

The following diagram shows a cuboid of square base of side x and height y. The volume of the cuboid is 125 cm^2



(a) Express y in term of x. [2]

(b) **Hence**, express the surface area S of the cuboid in terms of x. [2]

(c) Use $\frac{dS}{dx}$ to find the minimum value of S; justify your answer. [5]

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B. Paper 2 questions (LONG)

21. [Maximum mark: 12]

Let $f(x)=ax^3+bx^2+cx$

(a) Find the first and the second derivative of $f(x)$, in terms of a, b, c. [4]

(b) The graph

passes through the point $P(1,4)$

has a local maximum at P

has a point of inflection at $x=2$.

Write down three linear equations representing this information. [3]

(c) **Hence** find the values of a, b, c. [2]

(d) The function has a local minimum at $x=d$. Find the value of d and justify that it is a minimum. [3]

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22. [Maximum mark: 12]

The function f is given by $f(x) = \frac{\ln 2x}{x}$, $x > 0$.

(a) (i) Show that $f'(x) = \frac{1 - \ln 2x}{x^2}$.

Hence

(ii) prove that the graph of f can have only one local max or min point;

(iii) find the coordinates of the maximum point on the graph of f. [6]

(b) By showing that the second derivative $f''(x) = \frac{2\ln 2x - 3}{x^3}$ or otherwise,

find the coordinates of the point of inflexion on the graph of f. [6]

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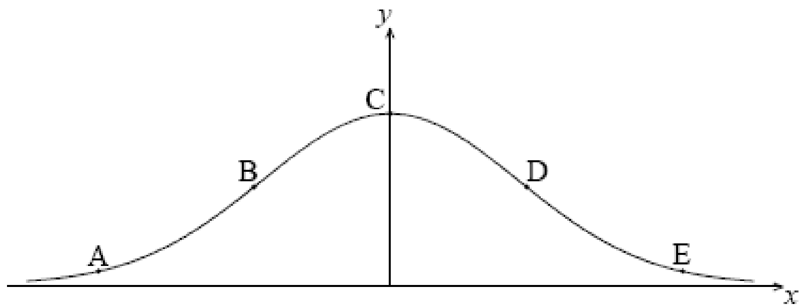
23. [Maximum mark: 13]

The following diagram shows the graph of $f(x) = e^{-x^2}$.



The points A, B, C, D and E lie on the graph of f. Two of these are points of inflexion.

(a) Identify the **two** points of inflexion. [2]

(b) (i) Find $f'(x)$.

(ii) Show that $f''(x) = (4x^2 - 2)e^{-x^2}$. [5]

(c) Find the x-coordinate of each point of inflexion. [4]

(d) Use the second derivative to show that one of these points is a point of inflexion. [2]

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24. [Maximum mark: 10]

Let the function f be defined by $f(x)=\frac{2}{1+x^3}$, $x\neq -1$.

(a) (i) Write down the equation of the vertical asymptote of the graph of f.

(ii) Write down the equation of the horizontal asymptote of the graph of f.

(iii) Sketch the graph of f in the domain $-3\leq x\leq 3$. [4]

(b) (i) Using the fact that $f'(x)=\frac{-6x^2}{(1+x^3)^2}$, show that the second derivative is $f''(x)=\frac{12x\left(2x^3-1\right)}{(1+x^3)^3}$.

(ii) Find the x-coordinates of the points of inflexion of the graph of f. [6]

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25. [Maximum mark: 14]

The function f is defined as $f(x)=(2x+1)e^{-x}$, $0\leq x\leq 3$. The point P(0, 1) lies on the graph of $f(x)$, and there is a maximum point at Q.

(a) Sketch the graph of $y=f(x)$, labelling the points P and Q. [3]

(b) (i) Show that $f'(x)=(1-2x)e^{-x}$.

(ii) Find the **exact** coordinates of Q. [7]

(c) The equation $f(x)=k$, where $k\in\mathbb{R}$, has two solutions. Write down the range of values of k. [2]

(d) Given that $f''(x)=e^{-x}(-3+2x)$, show that the curve of f has only one point of inflexion. [2]

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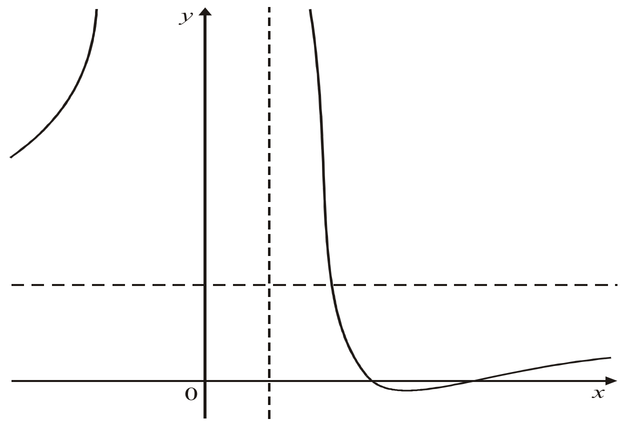
26. [Maximum mark: 10]

Consider the function f given by $f(x) = \dfrac{2x^2 - 13x + 20}{(x-1)^2}$, $x \neq 1$.

A part of the graph of f is given below (vertical and horizontal asymptotes are shown)



(a) Write down the **equation** of the vertical asymptote. [1]

(b) It is given that $f(100) = 1.91$, $f(-100) = 2.09$, $f(1000) = 1.99$

Evaluate $f(-1000)$ and write down the **equation** of the horizontal asymptote. [2]

(c) Show that $f'(x) = \dfrac{9x - 27}{(x-1)^3}$, $x \neq 1$. [3]

The second derivative is given by $f''(x) = \dfrac{72 - 18x}{(x-1)^4}$, $x \neq 1$.

(d) Using values of $f'(x)$ and $f''(x)$ explain why a minimum must occur at $x = 3$. [2]

(e) There is a point of inflexion on the graph. Write down the coordinates of this point. [2]

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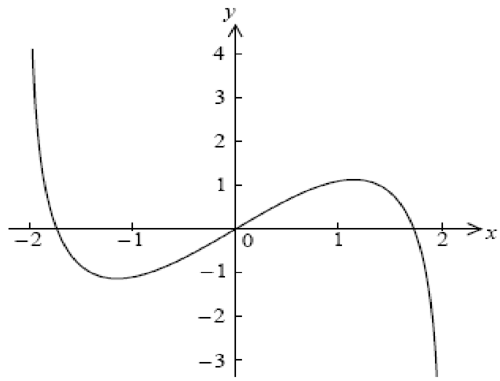
27. [Maximum mark: 13]

Consider $f(x) = x\ln(4-x^2)$, for $-2 < x < 2$. The graph of f is given below.



(a) Let P and Q be the stationary points on the curve of f.

(i) Find the x-coordinate of P and of Q.

(ii) Write down all values of k for which $f(x) = k$ has exactly two solutions. [5]

Let $g(x) = x^3\ln(4-x^2)$, for $-2 < x < 2$.

(b) Show that $g'(x) = \dfrac{-2x^4}{4-x^2} + 3x^2\ln(4-x^2)$. [4]

(c) By observing the graph of g', write down all values of w for which $g'(x) = w$ has exactly two solutions. [4]

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28. [Maximum mark: 13]

Let $f(x) = 3 + \dfrac{20}{x^2 - 4}$, for $x \neq \pm 2$. The graph of f is given below.

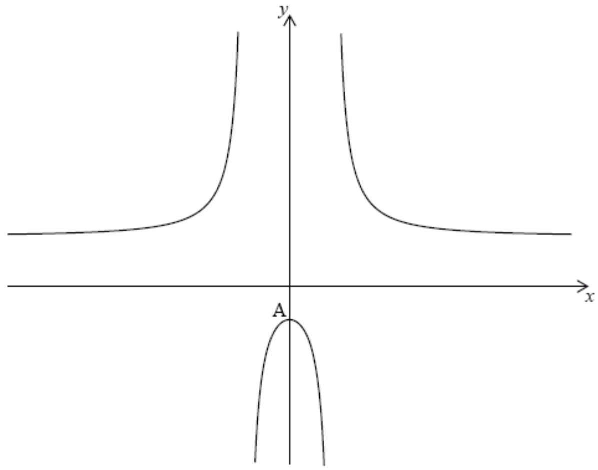

diagram not to scale

The y-intercept is at the point A.

(a) (i) Find the coordinates of A (ii) Show that $f'(x) = 0$ at A. [6]

(b) The second derivative of f is $f''(x) = \dfrac{40(3x^2 + 4)}{(x^2 - 4)^3}$. Use this to

(i) justify that the graph of f has a local maximum at A;

(ii) explain why the graph of f does **not** have a point of inflexion. [4]

(c) Describe the behaviour of the graph of f for large $|x|$. [1]

(d) Write down the range of f. [2]

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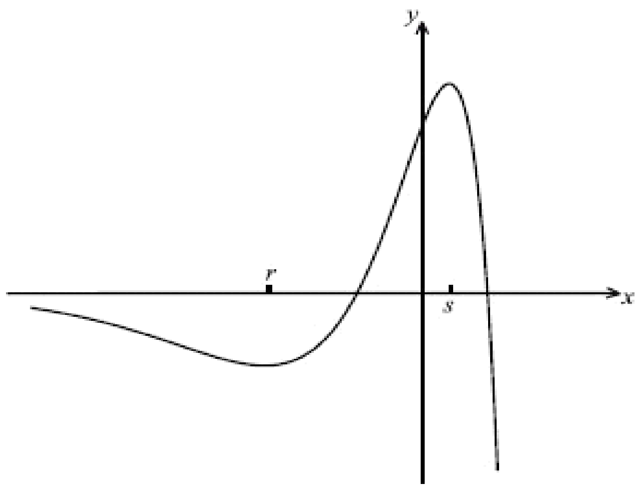
29. [Maximum mark: 14]

Let $f(x) = e^x(1-x^2)$.

(a) Show that $f'(x) = e^x(1-2x-x^2)$. [3]

Part of the graph of $y = f(x)$, for $-6 \le x \le 2$, is shown below. The x-coordinates of the local minimum and maximum points are r and s respectively.



(b) Write down the **equation** of the horizontal asymptote. [1]

(c) Write down the value of r and of s. [4]

(d) Show that the normal *L* to the curve of f at P(0, 1) has equation $x + y = 1$. [4]

(e) Find the coordinates of the points where the curve and the line *L* intersect. [2]

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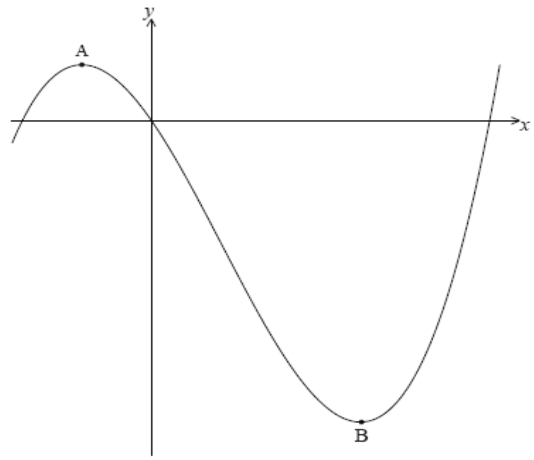
30. [Maximum mark: 12]

Let $f(x)=\frac{1}{3}x^3-x^2-3x$. Part of the graph of f is shown below.



There is a maximum point at A and a minimum point at B(3, –9).

(a) Find the coordinates of A. [6]

(b) Write down the coordinates of

(i) the image of B after reflection in the y-axis;

(ii) the image of B after translation by the vector $\begin{pmatrix}-2\\5\end{pmatrix}$;

(iii) the image of B after reflection in the x-axis followed by a horizontal stretch with scale factor $\frac{1}{2}$. [6]

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31. [Maximum mark: 12]

Let $f(x)=\dfrac{\cos x}{\sin x}$, for $\sin x \neq 0$.

(a) Use the quotient rule to show that $f'(x)=\dfrac{-1}{\sin^2 x}$. [4]

(b) Find $f''(x)$. [3]

In the following table, $f'\left(\dfrac{\pi}{2}\right)=p$ and $f'\left(\dfrac{\pi}{2}\right)=q$. The table also gives approximate values of $f'(x)$ and $f''(x)$ near $x=\dfrac{\pi}{2}$.

x	$\frac{\pi}{2}-0.1$	$\frac{\pi}{2}$	$\frac{\pi}{2}+0.1$
$f'(x)$	-1.01	p	-1.01
$f''(x)$	0.203	q	-0.203

(c) Find the value of p and of q. [3]

(d) Use information from the table to explain why there is a point of inflexion on the graph of f where $x=\dfrac{\pi}{2}$. [2]

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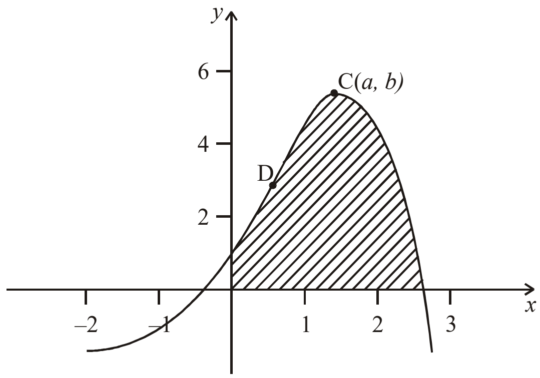
32. [Maximum mark: 12]

The diagram shows the graph of $y = e^x(\cos x + \sin x)$, $-2 \le x \le 3$. The graph has a maximum turning point at $\mathrm{C}(a, b)$ and a point of inflexion at D.



(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. [3]

(b) Find the **exact** value of a and of b. [4]

(c) Show that at D, $y = \sqrt{2}\mathrm{e}^{\frac{\pi}{4}}$. [5]

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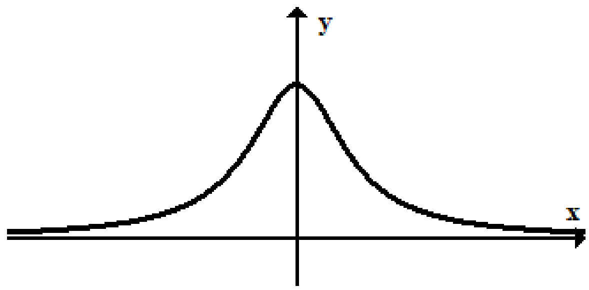
33. [Maximum mark: 12]

Part of the graph of $f(x)=\frac{1}{1+x^2}$ is shown below.



(a) Write down the equation of the horizontal asymptote of the graph of f. [1]

(b) Find $f'(x)$. [3]

(c) Show that the second derivative is given by $f''(x)=\frac{6x^2-2}{(1+x^2)^3}$. [4]

(d) Let A be the point on the curve of f where the gradient of the tangent is a maximum. Find the x-coordinate of A. [4]

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34. [Maximum mark: 11]

Consider the function $f(x)=e^{(2x-1)}+\dfrac{5}{2x-1}$, $x\neq\dfrac{1}{2}$.

(a) Sketch the curve of f for $-2\leq x\leq 2$, including any asymptotes. [3]

(b) Write down the equation of the vertical asymptote of f. [1]

(c) Find $f'(x)$. [4]

(d) (i) Write down the value of x at the minimum point on the curve of f.

(ii) The equation $f(x)=k$ has no solutions for $p\leq k<q$. Write down the value of p and of q. [3]

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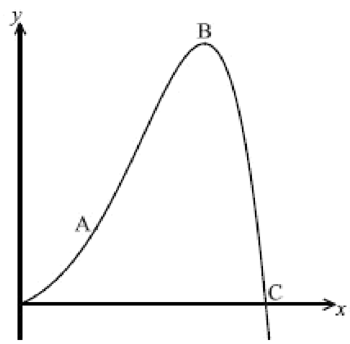
35. [Maximum mark: 12]

The function f is defined as $f(x) = e^x \sin x$, where x is in radians. Part of the curve of f is shown below.



There is a point of inflexion at A, and a local maximum point at B. The curve of f intersects the *x*-axis at the point C.

(a) Write down the x-coordinate of the point C. [2]

(b) (i) Find $f'(x)$.

(ii) Write down the value of $f'(x)$ at the point B. [4]

(c) Show that $f''(x) = 2e^x \cos x$. [2]

(d) (i) Write down the value of $f''(x)$ at A, the point of inflexion.

(ii) Hence, calculate the coordinates of A. [4]

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